

Walsh Functions and Spaces

Computational Mathematics and Statistics Seminar

Multiscale and Computation Seminar

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Fourier Series

When $f : [0, 1] \rightarrow \mathbb{R}$ has an absolutely convergent Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{2\pi i k x}, \quad \widehat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} dx.$$

When $f^{(\alpha)}$ has an absolutely convergent Fourier series for some $\alpha \in \mathbb{N}_0$ and $f^{(\beta)}$ periodic for all $\beta \in \{1, \dots, \alpha - 1\}$

$$f^{(\alpha)}(x) = \sum_{k \in \mathbb{Z}} \widehat{f^{(\alpha)}}(k) e^{2\pi i k x}, \quad \widehat{f^{(\alpha)}}(k) = (2\pi i k)^\alpha \widehat{f}(k).$$

Fourier Spaces

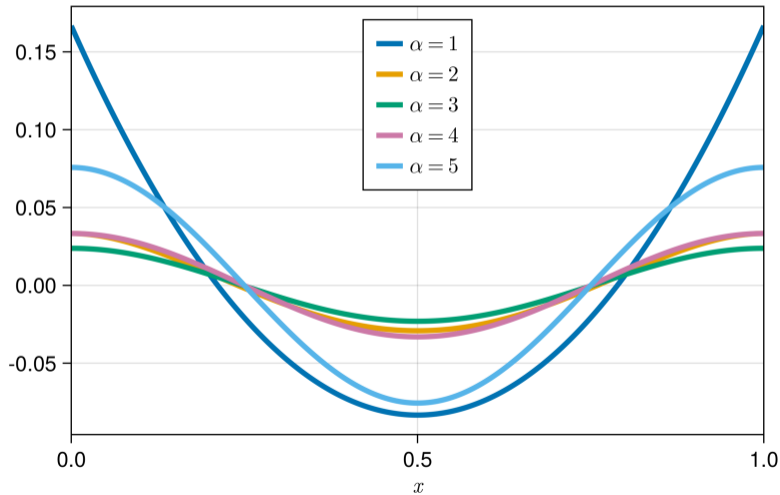
Let $\{x - y\} = (x - y) \bmod 1$ and let B_i denote the i^{th} Bernoulli polynomial.

$$\mathring{K}_\alpha(x, y) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi i k(x-y)}}{k^{2\alpha}} = \frac{(-1)^{\alpha+1} (2\pi)^{2\alpha}}{(2\alpha)!} B_{2\alpha}(\{x - y\}) = \mathring{K}_\alpha(\{x - y\}),$$

is the kernel of Sobolev RKHS \mathring{H}^α with $\alpha \in \mathbb{N}$ and

$$\langle f, g \rangle_\alpha^\circ = (-1)^\alpha (2\pi)^{-2\alpha} \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x) dx.$$

$$\mathring{K}_\alpha(x) = \frac{(-1)^{\alpha+1}(2\pi)^{2\alpha}}{(2\alpha)!} B_{2\alpha}(x)$$



Digitwise Operations

Prime base $b \geq 2$ expansion of $x \in [0, 1)$ is

$$x = .x_1x_2x_3 \cdots_b = \sum_{\ell \in \mathbb{N}} x_\ell b^{-\ell}, \quad \text{e.g.} \quad .375 = .011_2,$$

with digitwise addition (digitwise exclusive or in base $b = 2$)

$$x \oplus y := \sum_{\ell \in \mathbb{N}} ((x_\ell + y_\ell) \bmod b) b^{-\ell}, \quad \text{e.g.} \quad .375 \oplus .625 = .011_2 \oplus .101_2 = .110_2 = .75.$$

Similarly for $k \in \mathbb{N}_0$

$$k = \cdots k_2k_1k_0.0_b = \sum_{\ell \in \mathbb{N}_0} k_\ell b^\ell, \quad \text{e.g.} \quad 5 = 101_2,$$

$$k \oplus h := \sum_{\ell \in \mathbb{N}_0} ((k_\ell + h_\ell) \bmod b) b^\ell, \quad \text{e.g.} \quad 5 \oplus 6 = 101_2 \oplus 110_2 = 011_2 = 3.$$

Walsh Functions

Introduced for base $b = 2$ in [Walsh, 1923] with important results in [Fine, 1949].
Generalized to finite abelian group with a bijection in [Larcher et al., 1996].

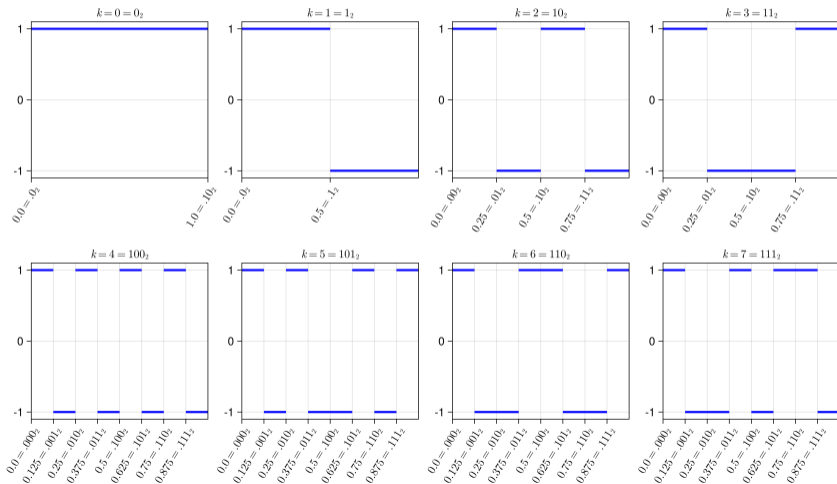
For $k \in \mathbb{N}_0$ with $\mathbf{k} = (k_0, k_1, \dots)$ and $x \in [0, 1)$ with $\mathbf{x} = (x_1, x_2, \dots)$,

$$\text{wal}_{\mathbf{k}}(x) = e^{2\pi i/b \sum_{\ell=0}^{\infty} k_{\ell} x_{\ell+1}} = e^{2\pi i/b \mathbf{k} \cdot \mathbf{x}}$$

e.g. for $b = 2$, $\text{wal}_6(.75) = (-1)^{(0,1,1) \cdot (1,1,0)} = -1$.

For any fixed b , $\{\text{wal}_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0\}$ is a complete orthonormal system in $\mathcal{L}_2([0, 1))$.
Notice similarity to complex exponential basis $\{e^{2\pi i k x} : k \in \mathbb{Z}\}$ for Fourier series.

$$b = 2 \text{ Walsh Functions } \text{wal}_k(x) = (-1)^{\sum_{\ell \in \mathbb{N}_0} k_\ell x_{\ell+1}}$$



Walsh Function Properties

For any $x, y \in [0, 1)$ and $k, h \in \mathbb{N}_0$ and $f \in \mathcal{L}_2([0, 1))$

1. $\text{wal}_k(x)\text{wal}_h(x) = \text{wal}_{k \oplus h}(x)$ and $\text{wal}_k(x)\text{wal}_k(y) = \text{wal}_k(x \oplus y)$

2.

$$\int_0^1 \text{wal}_k(x) dx = \begin{cases} 1, & k = 0 \\ 0, & k > 0 \end{cases}$$

3.

$$\int_0^1 f(\sigma) d\sigma = \int_0^1 f(x \oplus \sigma) d\sigma$$

4.

$$\sum_{k=0}^{b^a-1} \text{wal}_k(x) = \begin{cases} b^a, & a < \mathcal{I}(x) - 1 \\ 0, & \text{otherwise} \end{cases}$$

where $\mathcal{I}(x) = -\lfloor \log_b(x) \rfloor$ is the index first non-zero digit in the base b expansion
e.g. with $b = 2$ then $\mathcal{I}(.375) = \mathcal{I}(.011_2) = 2$.

Weight Function

Write $k \in \mathbb{N}$ as

$$k = \sum_{\ell=1}^{\#k} k_{a_\ell} b^{a_\ell}$$

with $a_1 > \dots > a_{\#k} \geq 0$ and $k_{a_\ell} \in \{1, \dots, b-1\}$.

Weight function for $\alpha \in \mathbb{N}_0$ has

$$\mu_\alpha(k) = \sum_{\ell=1}^{\min(\alpha, \#k)} (a_\ell + 1)$$

with $\mu_0(k) = \mu_\alpha(0) = 0$. μ sums indices of non-zero digits. For example, with $b = 2$

$$k = 13 = 1101_2 \quad \text{has} \quad (a_1, a_2, a_3) = (3, 2, 0)$$

$$\mu_1(k) = (3+1), \quad \mu_2(k) = (3+1) + (2+1), \quad \mu_3(k) = (3+1) + (2+1) + (0+1) = \mu_4(k) = \dots$$

Walsh Series of Smooth Functions

For $\alpha \geq 2$ the Sobolev RKHS H^α with inner product

$$\langle f, g \rangle_\alpha = \sum_{\beta=1}^{\alpha-1} \int_0^1 f^{(\beta)}(x) dx \int_0^1 g^{(\beta)}(x) dx + \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x) dx$$

has kernel

$$K_\alpha(x, y) = \sum_{\beta=1}^{\alpha-1} \frac{B_\beta(x) B_\beta(y)}{(\beta!)^2} + \overbrace{(-1)^{\alpha+1} \frac{B_{2\alpha}(\{x-y\})}{(2\alpha)!}}^{\hat{K}_\alpha(\{x-y\})/(2\pi)^{2\alpha}}.$$

[Dick, 2008, 2009] show that if $f \in H^\alpha$ then for $\hat{f}(k) = \int_0^1 f(x) \overline{\text{wal}_k(x)} dx$ we have

$$\sup_{k \in \mathbb{N}_0} \left| \hat{f}(k) \right| b^{\mu_\alpha(k)} < \infty \quad \text{i.e.} \quad \exists C_{f,\alpha} > 0 \quad \text{s.t.} \quad \left| \hat{f}(k) \right| \leq \frac{C_{f,\alpha}}{b^{\mu_\alpha(k)}}.$$

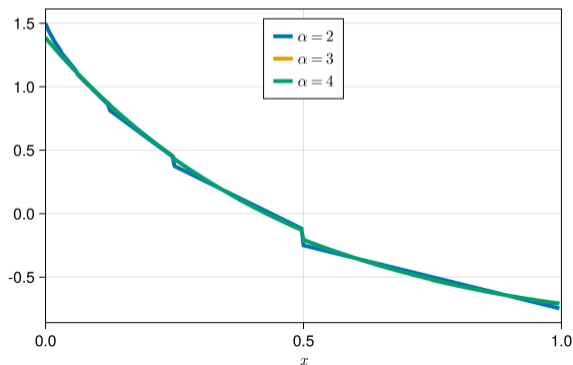
For the $\alpha = 1$ case see [Dick and Pillichshammer, 2005].

Digitally Shift Invariant Walsh Kernel

$H^\alpha \subset \tilde{H}^\alpha$ where \tilde{H}^α is an RKHS with kernel

$$\tilde{K}_\alpha(x, y) = \sum_{k \in \mathbb{N}} \frac{\text{wal}_k(x \ominus y)}{b^{\mu_\alpha(k)}} = \tilde{K}_\alpha(x \ominus y).$$

Below $\tilde{K}_\alpha(x)$ with $b = 2$ is shown. Discontinuities at $\{2^{-a} : a \in \mathbb{N}\}$ among others.



Research Connections for Multivariate Functions

Kernel matrix $K = (K(\mathbf{x}_t, \mathbf{x}_s))_{t,s=1}^N$ **of pairwise evaluations at** $\{\mathbf{x}_t\}_{t=1}^N \subset [0, 1]^d$ **is:**

- Generally dense and unstructured for $K_\alpha^\gamma(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^d [1 + \gamma_j K_{\alpha_j}(x_j, y_j)]$.
- **circulant** for $\mathring{K}_\alpha^\gamma(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^d [1 + \gamma_j \mathring{K}_{\alpha_j}(\{x_j - y_j\})]$, lattice $\{\mathbf{x}_t\}_{t=1}^N$.
- **block Toeplitz** for $\widetilde{K}_\alpha^\gamma(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^d [1 + \gamma_j \widetilde{K}_{\alpha_j}(x_j \ominus y_j)]$, digital net $\{\mathbf{x}_t\}_{t=1}^N$.

Kriging: Generally costs $\mathcal{O}(N^3)$. Costs $\mathcal{O}(N \log N)$ when K circulant or block Toeplitz.

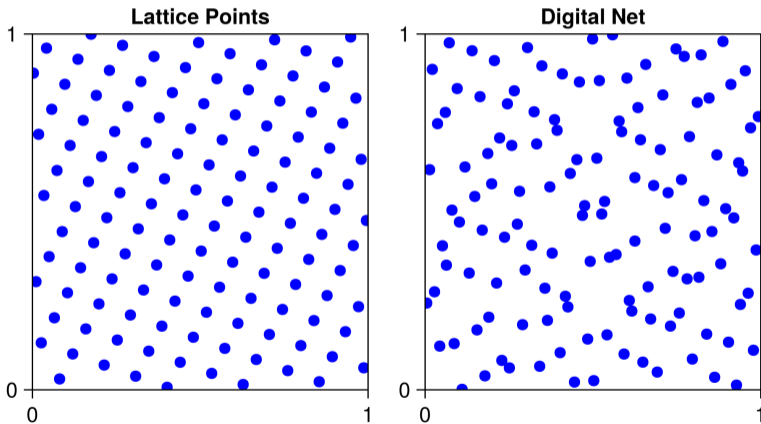
Quasi Monte Carlo Absolute Error

$$\left| \int_{[0,1]^d} f(\mathbf{x}) d\mathbf{x} - \frac{1}{N} \sum_{t=1}^N f(\mathbf{x}_t) \right|$$

- is $\mathcal{O}(N^{-\alpha+\delta})$ for f in RKHS of $\mathring{K}_\alpha^\gamma$ and certain lattice sequence $\{\mathbf{x}_t\}_{t=1}^N$.
- is $\mathcal{O}(N^{-\alpha+\delta})$ for f in RKHS of K_α^γ using certain digital sequences $\{\mathbf{x}_t\}_{t=1}^N$.

$\delta > 0$ arbitrary. Convergence rates independent of d when γ chosen judiciously.

Low Discrepancy (Quasi-Random) Point Sets



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